Study Guide

Permutations and Combinations

Use the **Basic Counting Principle** to determine different possibilities for the arrangement of objects. The arrangement of objects in a certain order is called a **permutation**. A **combination** is an arrangement in which order is *not* a consideration.

**Example 1**  Eight students on a student council are assigned 8 seats around a U-shaped table.

a. How many different ways can the students be assigned seats at the table?

Since order is important, this situation is a permutation. The eight students are taken all at once, so the situation can be represented as $P(8, 8)$.

$$P(8, 8) = \frac{8!}{(8-8)!} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40,320$$

There are 40,320 ways the students can be seated.

b. How many ways can a president and a vice-president be elected from the eight students?

This is a permutation of 8 students being chosen 2 at a time.

$$P(8, 2) = \frac{8!}{(8-2)!} = 8 \cdot 7 \cdot 6! = 56$$

There are 56 ways a president and vice-president can be chosen.

**Example 2**  The Outdoor Environmental Club consists of 20 members, of which 9 are male and 11 are female. Seven members will be selected to form an event-planning committee. How many committees of 4 females and 3 males can be formed?

Order is not important. There are three questions to consider.

How many ways can 3 males be chosen from 9?
How many ways can 4 females be chosen from 11?
How many ways can 3 males and 4 females be chosen together?

The answer is the product of the combinations $C(9, 3)$ and $C(11, 4)$.

$$C(9, 3) \cdot C(11, 4) = \frac{9!}{(9-3)!3!} \cdot \frac{11!}{(11-4)!4!} = \frac{9!}{6!3!} \cdot \frac{11!}{7!4!} = 84 \cdot 330 = 27,720$$

There are 27,720 possible committees.
**Practice**

**Permutations and Combinations**

1. A golf manufacturer makes irons with 7 different shaft lengths, 3 different grips, and 2 different club head materials. How many different combinations are offered?

2. A briefcase lock has 3 rotating cylinders, each containing 10 digits. How many numerical codes are possible?

3. How many 7-digit telephone numbers can be formed if the first digit cannot be 0 or 1?

**Find each value.**

4. \( P(10, 7) \)

5. \( P(7, 7) \)

6. \( P(6, 3) \)

7. \( C(7, 2) \)

8. \( C(10, 4) \)

9. \( C(12, 4) \cdot C(8, 3) \)

10. How many ways can the 4 call letters of a radio station be arranged if the first letter must be W or K and no letters can be repeated?

11. There are 5 different routes that a commuter can take from her home to her office. How many ways can she make a roundtrip if she uses different routes for coming and going?

12. How many committees of 5 students can be selected from a class of 25?

13. A box contains 12 black and 8 green marbles. How many ways can 3 black and 2 green marbles be chosen?

14. **Basketball** How many ways can a coach select a starting team of one center, two forwards, and two guards if the basketball team consists of three centers, five forwards, and three guards?
Permutations with Repetitions and Circular Permutations

For permutations involving repetitions, the number of permutations of \( n \) objects of which \( p \) are alike and \( q \) are alike is \( \frac{n!}{p!q!} \). When \( n \) objects are arranged in a circle, there are \( \frac{n!}{n} \), or \((n - 1)!\), permutations of the objects around the circle. If \( n \) objects are arranged relative to a fixed point, then there are \( n! \) permutations.

**Example 1** How many 10-letter patterns can be formed from the letters of the word *basketball*?

The ten letters can be arranged in \( P(10, 10) \), or \( 10! \), ways. However, some of these 3,628,800 ways have the same appearance because some of the letters appear more than once.

\[
\frac{10!}{2!2!2!} \quad \text{There are 2 as, 2 bs, and 2 ls in basket ball.}
\]

\[
\frac{10!}{2!2!2!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 453,600
\]

There are 453,600 ten-letter patterns that can be formed from the letters of the word *basketball*.

**Example 2** Six people are seated at a round table to play a game of cards.

a. Is the seating arrangement around the table a linear or circular permutation? Explain.

b. How many possible seating arrangements are there?

a. The arrangement of people is a circular permutation since the people form a circle around the table.

b. There are 6 people, so the number of arrangements can be described by \((6 - 1)!\).

\[(6 - 1)! = 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \text{ or } 120\]

There are 120 possible seating arrangements.
Permutations with Repetitions and Circular Permutations

How many different ways can the letters of each word be arranged?

1. members
2. annually
3. Missouri
4. concert

5. How many different 5-digit street addresses can have the digits 4, 7, 3, 4, and 8?

6. Three hardcover books and 5 paperbacks are placed on a shelf. How many ways can the books be arranged if all the hardcover books must be together and all the paperbacks must be together?

Determine whether each arrangement of objects is a linear or circular permutation. Then determine the number of arrangements for each situation.

7. 9 keys on a key ring with no chain
8. 5 charms on a bracelet with no clasp
9. 6 people seated at a round table with one person seated next to a door
10. 12 different symbols around the face of a watch

11. Entertainment Jasper is playing a word game and has the following letters in his tray: QUOUNNTAGGRA. How many 12-letter arrangements could Jasper make to check if a single word could be formed from all the letters?
Probability and Odds

The probability of an event is the ratio of the number of ways an event can happen to the total number of ways an event can and cannot happen.

**Example**

A bag contains 3 black, 5 green, and 4 yellow marbles.

**a. What is the probability that a marble selected at random will be green?**

The probability of selecting a green marble is written $P(\text{green})$. There are 5 ways to select a green marble from the bag and $3 + 4$, or 7, ways not to select a green marble. So, success $(s) = 5$ and failure $(f) = 7$. Use the probability formula.

$$P(\text{green}) = \frac{5}{5 + 7} \text{ or } \frac{5}{12} \hspace{1cm} P(s) = \frac{s}{s + f}$$

The probability of selecting a green marble is $\frac{5}{12}$.

**b. What is the probability that a marble selected at random will not be yellow?**

There are 8 ways not to select a yellow marble and 4 ways to select a yellow marble.

$$P(\text{not yellow}) = \frac{8}{4 + 8} \text{ or } \frac{2}{3} \hspace{1cm} P(f) = \frac{f}{s + f}$$

The probability of not selecting a yellow marble is $\frac{2}{3}$.

**c. What is the probability that 2 marbles selected at random will both be black?**

Use counting methods to determine the probability. There are $C(3, 2)$ ways to select 2 black marbles out of 3, and $C(12, 2)$ ways to select 2 marbles out of 12.

$$P(2 \text{ black marbles}) = \frac{C(3, 2)}{C(12, 2)}$$

$$= \frac{3!}{11!} \div \frac{12!}{10!} = \frac{1}{22} \text{ or } \frac{1}{22}$$

The probability of selecting 2 black marbles is $\frac{1}{22}$. 
Practice

Probability and Odds

A kitchen drawer contains 7 forks, 4 spoons, and 5 knives. Three are selected at random. Find each probability.

1. \( P(3 \text{ forks}) \)  
2. \( P(2 \text{ forks, 1 knife}) \)

3. \( P(3 \text{ spoons}) \)  
4. \( P(1 \text{ fork, 1 knife, 1 spoon}) \)

A laundry bag contains 5 red, 9 blue, and 6 white socks. Two socks are selected at random. Find each probability.

5. \( P(2 \text{ red}) \)  
6. \( P(2 \text{ blue}) \)

7. \( P(1 \text{ red, 1 blue}) \)  
8. \( P(1 \text{ red, 1 white}) \)

Sharon has 8 mystery books and 9 science-fiction books. Four are selected at random. Find each probability.

9. \( P(4 \text{ mystery books}) \)  
10. \( P(4 \text{ science-fiction books}) \)

11. \( P(2 \text{ mysteries, 2 science-fiction}) \)  
12. \( P(3 \text{ mysteries, 1 science-fiction}) \)

From a standard deck of 52 cards, 5 cards are drawn. What are the odds of each event occurring?

13. \( \text{5 aces} \)  
14. \( \text{5 face cards} \)

15. **Meteorology** A local weather forecast states that the chance of sunny weather on Wednesday is 70%. What are the odds that it will be sunny on Wednesday?
Probabilities of Compound Events

Example 1  Using a standard deck of playing cards, find the probability of drawing a king, replacing it, then drawing a second king.
Since the first card is returned to the deck, the outcome of the second draw is not affected by the first. The events are independent. The probability is the product of each individual probability.

\[ P(A \text{ and } B) = P(A) \cdot P(B) \]

Let \( A \) represent the first draw and \( B \) the second draw.

\[ P(A) = P(B) = \frac{4}{52} = \frac{1}{13} \]

4 kings
52 cards in a standard deck

\[ P(A \text{ and } B) = \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169} \]

The probability of selecting a king, replacing it, and then selecting another king is \( \frac{1}{169} \).

Example 2  What is the probability of selecting a yellow or a blue marble from a box of 5 green, 3 yellow, and 2 blue marbles?
A yellow marble and a blue marble cannot be selected at the same time. Thus, the events are mutually exclusive. Find the sum of the individual probabilities.

\[ P(\text{yellow or blue}) = P(\text{yellow}) + P(\text{blue}) \]

\[ = \frac{3}{10} + \frac{2}{10} \]

\[ P(\text{yellow}) = \frac{3}{10}, \quad P(\text{blue}) = \frac{2}{10} \]

\[ = \frac{5}{10} \text{ or } \frac{1}{2} \]

Example 3  What is the probability that a card drawn from a standard deck is either a face card or black?
The card drawn could be both a face card and black, so the events are mutually inclusive.

\[ P(\text{face card}) = \frac{12}{52} \]
\[ P(\text{black}) = \frac{26}{52} \]
\[ P(\text{face card and black}) = \frac{6}{52} \]
\[ P(\text{face card or black}) = \frac{12}{52} + \frac{26}{52} - \frac{6}{52} = \frac{32}{52} \text{ or } \frac{8}{13} \]
Practice

Probabilities of Compound Events

Determine if each event is independent or dependent. Then determine the probability.

1. the probability of drawing a black card from a standard deck of cards, replacing it, then drawing another black card

2. the probability of selecting 1 jazz, 1 country, and 1 rap CD in any order from 3 jazz, 2 country, and 5 rap CDs, replacing the CDs each time

3. the probability that two cards drawn from a deck are both aces

Determine if each event is mutually exclusive or mutually inclusive. Then determine each probability.

4. the probability of rolling a 3 or a 6 on one toss of a number cube

5. the probability of selecting a queen or a red card from a standard deck of cards

6. the probability of selecting at least three white crayons when four crayons are selected from a box containing 7 white crayons and 5 blue crayons

7. Team Sports Conrad tried out for both the volleyball team and the football team. The probability of his being selected for the volleyball team is \(\frac{4}{5}\), while the probability of his being selected for the football team is \(\frac{3}{4}\). The probability of his being selected for both teams is \(\frac{7}{10}\). What is the probability that Conrad will be selected for either the volleyball team or the football team?
Conditional Probabilities

The **conditional probability** of event \( A \), given event \( B \), is defined as \( P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} \), where \( P(B) \neq 0 \). In some situations, event \( A \) is a subset of event \( B \). In these situations, \( P(A \mid B) = \frac{P(A)}{P(B)} \), where \( P(B) \neq 0 \).

**Example**

Each of four boxes contains a red marble and a yellow marble. A marble is selected from each box without looking. What is the probability that exactly three red marbles are selected if the third marble is red?

Sample spaces and reduced sample spaces can be used to help determine the outcomes that satisfy a given condition.

The sample space is \( S = \{RRRR, RRRY, RYRR, RYRY, RYR, YYR, YRR, YRY, YRYR, YRYY, YYRR, YYRY, YYYR, YYYY\} \) and includes all of the possible outcomes of selecting 1 of the marbles from each of the 4 boxes. All of the outcomes are equally likely.

Event \( B \) represents the condition that the third marble is red.

\[ B = \{RRRR, RRRY, RYRR, RYRY, YRR, YYR, YRY, YRYR\} \]

\[ P(B) = \frac{8}{16} = \frac{1}{2} \]

Event \( A \) represents the condition that exactly three of the marbles are red.

\[ A = \{RRRY, RRYR, RYRR, YRRR\} \]

\((A \text{ and } B)\) is the intersection of \( A \) and \( B \).

\((A \text{ and } B) = \{RRRY, RRYR, YRRR\}\).

So, \( P(A \text{ and } B) = \frac{3}{16} \).

\[ P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} \]

\[ = \frac{\frac{3}{16}}{\frac{1}{2}} = \frac{3}{8} \]

The probability that exactly three marbles are red given that the third marble is red is \( \frac{3}{8} \).
Conditional Probabilities

Find each probability.

1. Two number cubes are tossed. Find the probability that the numbers showing on the cubes match, given that their sum is greater than 7.

2. A four-digit number is formed from the digits 1, 2, 3, and 4. Find the probability that the number ends in the digits 41, given that the number is odd.

3. Three coins are tossed. Find the probability that exactly two coins show tails, given that the third coin shows tails.

A card is chosen from a standard deck of cards. Find each probability, given that the card is red.

4. P(diamond)  
5. P(six of hearts)

6. P(queen or 10)  
7. P(face card)

A survey taken at Stirers High School shows that 48% of the respondents like soccer, 66% like basketball, and 38% like hockey. Also, 30% like soccer and basketball, 22% like basketball and hockey, and 28% like soccer and hockey. Finally, 12% like all three sports.

8. If Meg likes basketball, what is the probability that she also likes soccer?

9. If Jaime likes soccer, what is the probability that he also likes hockey and basketball?

10. If Ashley likes basketball, what is the probability that she also likes hockey?

11. If Brett likes soccer, what is the probability that he also likes basketball?